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The steady motions of a satellite with a two-degree-of-freedom powered gyroscope in a central gravitational field and their stability☆

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ABSTRACT

The positions of relative equilibrium of a satellite carrying a two-degree-of-freedom powered gyroscope, in the axes of the framework of which only dissipative forces can act, are investigated within the limits of a restricted circular problem. For the case when the "satellite - gyroscope" system possesses the property of a gyrostat and the axis of the gyroscope frame is directed parallel to one of the principal central axes of inertia of the satellite, all the equilibrium positions are found as a function of the magnitude of the angular momentum of the rotor. It is established that the minimum number of equilibrium positions is equal to 32 and, in certain ranges of values of the system parameters, it can reach 80. All the positions satisfying the sufficient conditions for stability are also determined. The number of them is either equal to 4 or 8 depending on the values of the system parameters.

Up to the present time, the problem of the steady motions (equilibrium positions with respect to an orbital basis) of satellites carrying powered rotors has been fairly fully investigated (Refs 1–4, 9, 10, etc.). The steady motions of satellites carrying powered gyroscopes have been studied to a lesser extent. In the few papers on this topic, [5–8] only specific steady motions of a satellite with two-degree-of-freedom and three-degree-of-freedom gyroscopes have been investigated.

In this paper, the problem of determining the set of all steady motions of a satellite, carrying a two-degree-of-freedom powered gyroscope, in a central gravitational field is formulated. A similar problem has been solved previously [11,12] for the case of a uniform external field.

1. Formulation of the problem

A satellite consisting of a load-carrying rigid body (a housing) and a two-degree-of-freedom powered gyroscope is considered. It is assumed that the rotor of the gyroscope rotates at a constant angular velocity with respect to the frame, the angle of rotation of the frame is unrestricted and that the axis of the rotor is orthogonal to the axis of the frame (Fig. 1). We will specify the direction of the axis of suspension of the frame with respect to the housing of the satellite by the unit vector $\mathbf{s}$ and we will denote the angle of rotation of the frame by x. We will specify the current position of the axis of the rotor by the unit vector $\mathbf{h}(x)$, which is directed in the sense of the rotation of the rotor. The angular momentum of the characteristic rotation of the rotor will then be given by the expression $\tilde{\mathbf{H}} = \tilde{H}\mathbf{h}$, where $\tilde{H} = \text{const} > 0$.

The steady motions of the satellite are analysed within the limits of a bounded circular problem, that is, assuming that the centre of mass of the satellite moves in a circular Kepler orbit. The mutually orthogonal unit vectors $\mathbf{r}$, $\mathbf{n}$ and $\boldsymbol{\tau} = \mathbf{n} \times \mathbf{r}$, directed along the radius of the orbit, along the normal to the plane of the orbit, and along a tangent to the orbit respectively serve as the orbital basis. We will denote the angular velocity of the orbital basis by Ω and the "reduced" angular momentum of the rotor by $\mathbf{H} = H\mathrm{h}$, where $H = \tilde{H}/\Omega = \text{const} > 0$.

The stationary points of the transformed potential energy of the system, which, in the general case when no constraints are imposed on the geometry of the masses of the gyroscope, is given by the formula

$$W = \Omega^2(-\mathbf{n}^T\mathbf{J}\mathbf{n}^2/2 - \mathbf{n}^T\mathbf{H} + (3\mathbf{r}^T\mathbf{J}\mathbf{r} - \text{tr}\mathbf{J})/2) + U(x) \tag{1.1}$$

correspond [1,2] to the positions of relative equilibrium of a satellite in a circular orbit.

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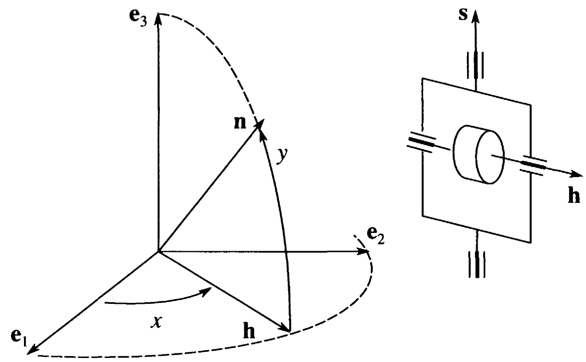

Fig. 1.

Here, $U(x)$ is the energy of the potential forces acting in the axes of the frame and the inertia tensor of the system $\mathbf{J}=\mathbf{J}(x)$ and the vectors $\mathbf{n}$ and $\mathbf{r}$ are written in a basis which has its origin at the current centre of mass of the satellite and an unchanged orientation with respect to the housing of the satellite.

When the gyroscope is statically balanced about the axis of the frame (the centre of mass of the gyroscope lies in the axis of the frame $\mathbf{s}$), the centre of mass of the satellite is fixed with respect to the housing and the term tr$\mathbf{J}$ will be a constant quantity and can be omitted in expression (1.1). With the additional condition of dynamic symmetry of the gyroscope about the $\mathbf{s}$ axis, the satellite will be a gyrostat, that is, its inertial tensor $\mathbf{J}$ will be invariant.

We will next assume that the satellite is a gyrostat and, among its principal central moments of inertia A, B and C, none is different from the other. It is also assumed that only the moments of dissipative forces can act in the axes of the frame and that there are no potential forces, that is, $U(x)\equiv 0$. The special case when the axis of the frame of the gyroscope is set parallel to one of the principal central axes of inertia of the satellite will be investigated in detail.

2. Steady motions

Since there are no potential forces in the axes of the frame of the gyroscope and the system is a gyrostat, its equilibrium positions can be defined as the stationary points of the function

$$W = -\mathbf{n}^T\mathbf{J}\mathbf{n}/2 - \mathbf{n}^T\mathbf{H} + 3\mathbf{r}^T\mathbf{J}\mathbf{r}/2 \tag{2.1}$$

where the inertia tensor $\mathbf{J}$ and the vectors $\mathbf{n}$ and $\mathbf{r}$ are written in the basis of the principal central axes of inertia of the system $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ associated with the housing.

Since the vectors $\mathbf{n}$ and $\mathbf{r}$ are unit vectors and mutually orthogonal, then, using a Lagrangian function with the multiplies

$$L = W + \lambda_1\mathbf{n}^T\mathbf{n}/2 + \lambda_2\mathbf{r}^T\mathbf{r}/2 + \lambda_3\mathbf{n}^T\mathbf{r} \tag{2.2}$$

we obtain the following system of equations for the equilibrium positions

$$\partial L/\partial\mathbf{n} = -\mathbf{J}\mathbf{n} - \mathbf{H} + \lambda_1\mathbf{n} + \lambda_3\mathbf{r} = 0 \tag{2.3}$$

$$\partial L/\partial\mathbf{r} = 3\mathbf{J}\mathbf{r} + \lambda_2\mathbf{r} + \lambda_3\mathbf{n} = 0 \tag{2.4}$$

$$\partial L/\partial x = \mathbf{s}^T(\mathbf{n}\times\mathbf{H}) = 0 \tag{2.5}$$

The mutual orthogonality of the axis of the frame and the axis of the rotor also give the equation

$$\mathbf{s}^T\mathbf{h} = 0 \tag{2.6}$$

Note that Eq. (2.5) describes the equilibrium condition of a gyroscope with respect to the housing of the satellite, and Eqs. (2.3) and (2.4) are equivalent to the single equation

$$\mathbf{n}\times(\mathbf{J}\mathbf{n}+\mathbf{H}) = 3\mathbf{r}\times\mathbf{J}\mathbf{r}$$

which describes the equilibrium condition of a satellite with a rotor with respect to the orbital basis.

The "direct" problem consists of finding the values of the angle x (the position of the rotor axis $\mathbf{h}$) and the positions of the vectors $\mathbf{r}$ and $\mathbf{n}$ as a function of the magnitude of the reduced angular momentum of the rotor H from system (2.3)–(2.6).

We introduce the following scalar functions of the vector $\mathbf{r}$

$$f = \mathbf{r}^T\mathbf{J}\mathbf{r}, \quad g = \sqrt{(\mathbf{J}\mathbf{r})^2 - f^2} = \sqrt{\mathbf{r}^T\mathbf{J}^2\mathbf{r} - (\mathbf{r}^T\mathbf{J}\mathbf{r})^2} \tag{2.7}$$

From Eq. (2.4), we find

$$\lambda_2 = -3f, \quad \lambda_3 = \sigma 3g, \quad \mathbf{n} = \sigma(f\mathbf{r} - \mathbf{J}\mathbf{r})/g; \quad \sigma = \pm 1 \tag{2.8}$$

Taking account of relations (2.8), the vector **H** is determined from Eq. (2.3) by the formula

$$\mathbf{H} = \sigma((\mathbf{J}^2\mathbf{r} - f\mathbf{J}\mathbf{r})/g + 3g\mathbf{r} - \lambda_1(\mathbf{J}\mathbf{r} - f\mathbf{r})/g) \tag{2.9}$$

Substituting expression (2.9) into Eqs. (2.6) and (2.5), we obtain the following equations

$$\mathbf{s}^T\mathbf{J}^2\mathbf{r} - (f+\lambda_1)\mathbf{s}^T\mathbf{J}\mathbf{r} + (3g^2 + \lambda_1 f)\mathbf{s}^T\mathbf{r} = 0 \tag{2.10}$$

$$\mathbf{s}^T(\mathbf{J}\mathbf{r}\times\mathbf{J}^2\mathbf{r}) - f\mathbf{s}^T(\mathbf{r}\times\mathbf{J}^2\mathbf{r}) + (f^2 - 3g^2)\mathbf{s}^T(\mathbf{r}\times\mathbf{J}\mathbf{r}) = 0 \tag{2.11}$$

Equation (2.11) only contains the variable **r** and describes the possible values of the vector **r** for steady motions. The corresponding values of the vectors **n** and **H** can be determined as two-valued functions of the vector **r** using formulae (2.8) and (2.9), expressing the factor λ_1 from Eq. (2.10). Hence, the problem of determining of the set of all the steady motions of the system being considered can be reduced to the problem of finding the set of all solutions of Eq. (2.11). The solution of the "direct" problem can be found by of an analysis of the behaviour of the quantity H in the set of solutions (2.11) using formulae (2.8)–(2.10).

Note that, if the vector **r** is directed along one of the principal axes of inertia, then indeterminacies of the form 0/0 occur in formulae (2.8) and (2.9). Hence, in the case of such solutions of Eq. (2.11), it is necessary to find the values of the vectors **n** and **H** directly from system (2.3)–(2.6).

We will next consider the case when the axis of the frame of the gyroscope is parallel to one of the principal central axes of inertia of the system. To be specific, we put $\mathbf{S} = \mathbf{e}_3$. The angular momentum of the rotor is then determined in terms of the angle x, measured from the $\mathbf{e}_1$ axis, by the expression

$$\mathbf{H} = H\mathbf{h} = H(\mathbf{e}_1\cos x + \mathbf{e}_2\sin x) \tag{2.12}$$

By virtue of Eq. (2.5), the vectors $\mathbf{e}_3$, **h** and **n** lie in one plane. Therefore, denoting the angle between the vectors **h** and **n** by y (Fig. 1), we obtain the following representation for the vector **n**

$$\mathbf{n} = \mathbf{h}\cos y + \mathbf{e}_3\sin y = (\mathbf{e}_1\cos x + \mathbf{e}_2\sin x)\cos y + \mathbf{e}_3\sin y \tag{2.13}$$

Denoting the direction cosines of the vector **r** with axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ by r_1, r_2, r_3, we find

$$\mathbf{s}^T(\mathbf{r}\times\mathbf{J}\mathbf{r}) = (B-A)r_1r_2, \quad \mathbf{s}^T(\mathbf{J}\mathbf{r}\times\mathbf{J}^2\mathbf{r}) = AB(B-A)r_1r_2$$
$$\mathbf{s}^T(\mathbf{r}\times\mathbf{J}^2\mathbf{r}) = (B^2 - A^2)r_1r_2 \tag{2.14}$$

After substituting expressions (2.14) into Eq. (2.11), we obtain the expression

$$r_1r_2((f-A)(f-B) - 3g^2) = 0 \tag{2.15}$$

which decomposes into two equations:

$$r_1r_2 = 0 \tag{2.16}$$

$$(f-A)(f-B) = 3g^2 \tag{2.17}$$

The vectors **r**, lying in one of the principal planes of inertia $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_2$, $\mathbf{e}_3$, are the solutions of Eq. (2.16). Omitting the intermediate calculations, we present the equilibrium positions found from system (2.2)–(2.6), corresponding to the solutions of Eq. (2.16). Here, we use the symbols δ_1, δ_2, δ_3 to represent multivalued solutions, each of which can take two values ± 1 and, also, the following notation for the difference between the moments of inertia

$$a = A - C, \quad b = B - C \tag{2.18}$$

Two groups of equilibrium positions correspond to the solutions of Eq. (2.16). The first group consists of the "direct" equilibrium positions (when the vectors of the orbital basis are directed along the principal axes of inertia of the system) which exist for any value $H \in (0, +\infty)$ and are described by the formulae

$$\mathbf{r} = \delta_1\mathbf{e}_3, \quad \mathbf{n} = \delta_2\mathbf{e}_k, \quad \mathbf{h} = \delta_3\mathbf{e}_k; \quad k = 1, 2 \tag{2.19}$$

$$\mathbf{r} = \delta_1\mathbf{e}_{3-k}, \quad \mathbf{n} = \delta_2\mathbf{e}_k, \quad \mathbf{h} = \delta_3\mathbf{e}_k; \quad k = 1, 2 \tag{2.20}$$

The following "skew" equilibrium positions constitute the second group

$$\mathbf{r} = \delta_1\mathbf{e}_{3-k}, \quad \mathbf{h} = \delta_3\mathbf{e}_k, \quad \mathbf{n} = \delta_3\mathbf{e}_k\cos y + \delta_2\mathbf{e}_3|\sin y|$$
$$\cos y = -H/a_k, \quad H \in (0, |a_k|]; \quad k = 1, 2 \tag{2.21}$$

$$\mathbf{h} = \delta_3\mathbf{e}_k, \quad \mathbf{r} = \delta_1(\delta_3\mathbf{e}_3\cos y - \delta_2\mathbf{e}_k|\sin y|), \quad \mathbf{n} = \delta_3\mathbf{e}_k\cos y + \delta_2\mathbf{e}_3|\sin y|$$
$$\cos y = -H/(4a_k), \quad H \in (0, 4|a_k|]; \quad k = 1, 2 \tag{2.22}$$

Here, $a_1 = a$, $a_2 = b$.

Relations (2.19) and (2.20) define 32 equilibrium positions. In their turn, each of relations (2.21) and (2.22) defines 16 equilibrium positions, the domains of existence of which are bounded by half-intervals of the values of H. Solutions (2.21) and (2.22) bifurcate from the corresponding solutions of (2.19) and (2.20) at the right-hand boundary points of the above-mentioned half-intervals. For example, the solutions of (2.21) when $k=1$ branch off from the four solutions of (2.20) when $k=1$ for which $\delta_2=-\delta_3 \operatorname{sign} a$ at the point $H=|a|$.

For values of H close to zero, the number of equilibrium positions described by formulae (2.19)–(2.22) is equal to 64 and, as the parameter H increases, it reduces by 8 units on passing through each branching point, attaining a minimum value of 32.

Note that, for all the equilibrium positions found above, the axis of the rotor is parallel to one of the principal axes of inertia of the satellite $\mathbf{e}_1$ or $\mathbf{e}_2$.

We will now investigate the solutions of Eq. (2.17) and the equilibrium positions corresponding to them. We define the vector $\mathbf{r}$ using spherical coordinates

$$\mathbf{r} = \mathbf{e}_3\cos\theta + \sin\theta(\mathbf{e}_1\cos\psi + \mathbf{e}_2\sin\psi), \quad 0 \le \theta \le \pi \tag{2.23}$$

and then obtain

$$f = C\cos^2\theta + \sin^2\theta(A\cos^2\psi + B\sin^2\psi)$$

$$g^2 = C^2\cos^2\theta + \sin^2\theta(A^2\cos^2\psi + B^2\sin^2\psi) - f^2$$

Substituting these expressions into Eq. (2.17), we arrive at a quadratic equation in the unknown $\operatorname{ctg}^2\theta$ with coefficients which depend on ψ. It can be shown that this equation only has real solutions when the inequality $ab\sin^2 2\psi \ge 0$ is satisfied. In this case, the solution has the form

$$\operatorname{ctg}^2\theta = \left(F + \sqrt{F^2 + 4ab(a-b)^2\sin^2 2\psi}\right)/(2ab) \tag{2.24}$$

where

$$F = 4(a^2\cos^2\psi + b^2\sin^2\psi) - ab$$

and, for each angle ψ, it defines two values of the angle θ in the range $[0,\pi]$ and the set of solutions for the vector $\mathbf{r}$ consists of two curves on a unit sphere which are symmetrical about the $\mathbf{e}_1$, $\mathbf{e}_2$ plane and are closed around the $\mathbf{e}_3$ axis. At the points $\sin 2\psi = 0$, the solutions of (2.24) intersect with the solutions of Eq. (2.16). It therefore follows from the inequality $ab\sin^2 2\psi \ge 0$ that Eq. (2.17) gives new solutions which are different from the solutions of Eq. (2.16) only when the condition

$$ab = (A-C)(B-C) > 0 \tag{2.25}$$

is satisfied, that is, when the axis of the frame is set parallel to the axis of the greatest or the axis of the smallest moment of inertia of the system. The set of equilibrium positions obtained here is described by formulae (2.24), (2.8) and (2.9) and is a four-valued one-parameter set, the parameter of which is the angle ψ.

We will now consider the "direct" problem, that is, we will find how the equilibrium positions, defined by Eq. (2.17), depends on the magnitude of the angular momentum of the rotor H.

We express the functions f and g^2 in terms of the direction cosines of the vector $\mathbf{r}$:

$$f = Ar_1^2 + Br_2^2 + Cr_3^2, \quad g^2 = A^2r_1^2 + B^2r_2^2 + C^2r_3^2 - f^2 \tag{2.26}$$

Whence, using Eq. (2.17) and the identity $r_1^2 + r_2^2 + r_3^2 = 1$, we find the expressions for the squares of the direction cosines in terms of the function f:

$$3a(a-b)r_1^2 = (f-B)(4(f-C)-a)$$

$$3b(b-a)r_2^2 = (f-A)(4(f-C)-b)$$

$$3abr_3^2 = 12g^2 = 4(f-A)(f-B) \tag{2.27}$$

In turn, projecting the equality (2.9) onto the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes, we obtain

$$g^2H_1^2 = r_1^2(f-A)^2(A-\lambda_1+B-f)^2$$

$$g^2H_2^2 = r_2^2(f-B)^2(A-\lambda_1+B-f)^2 \tag{2.28}$$

The Lagrange multiplier λ_1 is determined from Eq. (2.10) by the formula

$$\lambda_1 = C + 3g^2/(C-f) \tag{2.29}$$

Using relations (2.29), (2.27) and (2.17), we obtain from Eqs. (2.28) a system in the variable $z=f-C$

$$(H_1^2(a-b) - 4ab^2)z^2 + 5a^2b^2z - a^3b^2 = 0$$

$$(H_2^2(b-a) - 4ba^2)z^2 + 5a^2b^2z - b^3a^2 = 0 \tag{2.30}$$

Taking account of the equality $H_1^2 + H_2^2 = H^2$, we transform this system to the following form

$$(H^2 + 4ab)z^2 = a^2b^2 \tag{2.31}$$

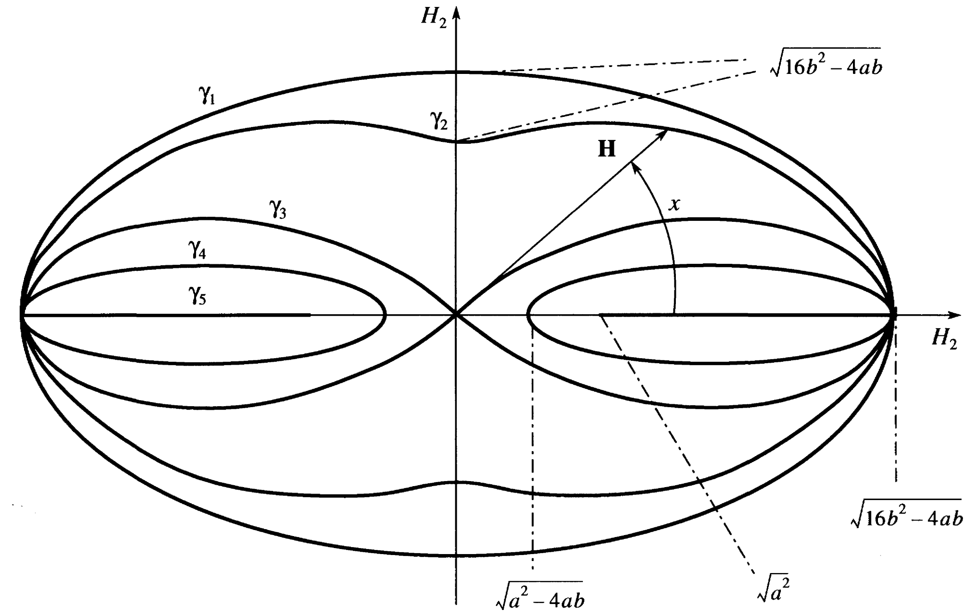

Fig. 2.

$$5(H^2+4ab)z = bH_1^2+aH_2^2+4(a+b)ab \tag{2.32}$$

It follows from Eqs (2.32) and inequality (2.25) that, for the solutions being considered, the sign of the variable z is the same as the sign of the numbers a and b. The expression

$$z = f-C = (ab/\sqrt{H^2+4ab})\operatorname{sign}a \tag{2.33}$$

therefore follows from Eq. (2.31), and, when this is substituted into Eq. (2.32), we obtain the equation

$$H_1^2+\varepsilon H_2^2+4ab(\varepsilon+1) = 5|a|\sqrt{H^2+4ab}, \quad \varepsilon = a/b>0 \tag{2.34}$$

It defines a fourth-order curve which is symmetrical about the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes. By virtue of condition (2.25), the principal axes $\mathbf{e}_1$ and $\mathbf{e}_2$ can be chosen such that the inequality $a/b>1$ is satisfied. The behaviour of the curve as a function of the relation between the parameters a and b will then have the form shown in Fig. 2.

The curves γ_1 and γ_2 correspond to the case when $a/b<4$, where the parameters satisfy the inequality $a/b<8/5$ in the case of the curve γ_1 and the inequality $a/b>8/5$ in the case of the curve γ_2. When $a/b=4$, the curve is transformed into a "figure eight" γ_3 while the two isolated symmetric curves γ_4 correspond to the case when $a/b>4$. When $b\to 0$, the curves γ_4 degenerate into two segments γ_5. All the curves are "stretched" along the $\mathbf{e}_1$ axis which, in the case when $a>0$, $b>0$, will be the axis of greatest moment of inertia and, in the case when $a<0$, $b<0$, it will be the axis of smallest moment of inertia.

The formula

$$\cos^2 x = \frac{\varepsilon H^2+4ab(\varepsilon+1)-5|a|\sqrt{H^2+4ab}}{(\varepsilon-1)H^2} \tag{2.35}$$

follows from Eq. (2.34). This formula determines the dependence of the angle of rotation of the frame x on the magnitude of the angular momentum of the rotor H for the equilibrium positions being considered. The range of values of the angular momentum H, for which solutions (2.35) are defined, is given by the inequalities

$$\sqrt{16b^2-4ab} \le H \le \sqrt{16a^2-4ab} \text{ when } a/b\le 4 \tag{2.36}$$

$$\sqrt{a^2-4ab} \le H \le \sqrt{16a^2-4ab} \text{ when } a/b\ge 4 \tag{2.37}$$

For each internal point II from the above-mentioned range, formula (2.35) defines four values of the angle x.

We will now find the dependence of the vectors **n** and **r** on the magnitude of the parameter H for the set of equilibrium positions being considered. From relations (2.8), we obtain

$$\mathbf{n}^T\mathbf{J}\mathbf{n} = (A(f-A)^2r_1^2+B(f-B)^2r_2^2+C(f-C)^2r_3^2)/g^2 \tag{2.38}$$

Substituting expressions (2.27) here and taking account of Eq. (2.17), we obtain

$$\mathbf{n}^T\mathbf{J}\mathbf{n} = -4(f-C)-f+A+B \tag{2.39}$$

In turn, it follows from Eq. (2.3), when account is taken of the equalities (2.29) and (2.39), that

$$\mathbf{n}^T\mathbf{H} = H\cos y = -\mathbf{n}^T\mathbf{J}\mathbf{n} + \lambda_1 = 4(f-C) - ab/(f-C)$$

Hence, substituting expression (2.33), we obtain the solution

$$\cos y = (H/\sqrt{H^2+4ab})\operatorname{sign}(C-A) \tag{2.40}$$

which defines two values of the angle for each value of H. Since the vector $\mathbf{n}$ is defined by formula (2.13) in terms of the angles x and y and the solution $x(H)$ is four-valued, eight values of the vector $\mathbf{n}$ correspond to each value of H. In turn, the vector $\mathbf{r}$ is found from Eq. (2.3) in terms of H, x and y (in terms of $\mathbf{H}$ and $\mathbf{n}$) as a two-valued function according to the formula

$$\mathbf{r} = \pm(\mathbf{J}\mathbf{n} + \mathbf{H} - \mathbf{n}(\mathbf{n}^T\mathbf{J}\mathbf{n} + \mathbf{n}^T\mathbf{H}))/|\mathbf{J}\mathbf{n} + \mathbf{H} - \mathbf{n}(\mathbf{n}^T\mathbf{J}\mathbf{n} + \mathbf{n}^T\mathbf{H})| \tag{2.41}$$

Hence, 16 different equilibrium positions correspond to each point of H from the range of existence of the solutions being considered.

It follows from the above analysis of the dependence of all the equilibrium positions found on the magnitude of H that the total number of equilibrium positions can reach 80 and that the minimum number of equilibrium positions is equal to 32. When $H=0$, the angle x becomes a cylindrical coordinate and it can take any value at the equilibrium positions of the system. At the same time, the number of equilibrium orientations of the housing of the satellite with respect to the orbital basis becomes equal to 24.

Note that the set of equilibrium positions for the problem considered with a two-degree-of-freedom gyroscope can be interpreted as a subset of the solutions of the analogous problem for a satellite with a rotor, the axis of which is parallel to the principal plane of inertia $\mathbf{e}_1$, $\mathbf{e}_2$.[4,9,10] In the case of a problem with a rotor, the angle x is a fixed parameter and the equilibrium positions are determined from system (2.3), (2.4). By determining the solutions of system (2.3), (2.4) for each value $x \in [0, 2\pi]$ and different values of H and subsequently picking out only those of them which satisfy Eq. (2.5), it is possible to obtain all the equilibrium positions corresponding to the angle x for the problem with a gyroscope.

For example, for an angle $x=0$, we shall have $\mathbf{H}=H\mathbf{e}_1$. The equilibrium positions of a satellite with a rotor corresponding to this case are defined by six groups of solutions with four solutions in each group.[9] Picking out from these solutions only those that satisfy Eq. (2.15), we obtain solutions described by relations (2.19)–(2.22) for $\delta_3=1$, $k=1$. Similarly, it is possible to obtain all the other equilibrium positions (2.19)–(2.22) by considering the angles $x=\pi/2$, π, $3\pi/2$.

The remaining solutions, which are described by formulae (2.35), (2.40) and (2.41), are determined from the solutions for the problem with a rotor when $x \neq k\pi/2$, where k is an integer. In this problem, the equilibrium positions of the satellite are defined by three groups of solutions,[10] but condition (2.5) is only satisfied in the case of one of these groups and only for values of H which satisfy equality (2.35).

It should be noted that it is not advisable to use the above correlation to find the equilibrium positions for the problem with a gyroscope directly from the solutions of the corresponding problem with a rotor. The solutions for the problem with a rotor, the axis of which is parallel to the principal plane of inertia, are determined from equations of the fourth degree. At the same time, by taking account of condition (2.5) in the final stage, as was done in this paper, one obtains simpler equations, the solutions of which describe the whole set of equilibrium positions and are written in explicit form.

3. Analysis of the stability of the equilibrium positions

We will now find the equilibrium positions of the system which satisfy sufficient conditions for stability (which possess secular stability). These positions will be stable regardless of the presence or absence of dissipation in the axes of the frame of the gyroscope.

Equilibrium positions in which the function (2.1) has a strictly local minimum possess secular stability. We will determine these solutions in two stages. We will first find a solution which gives a strict minimum of the function (2.1) with respect to the angle x for a fixed orientation, that is, for fixed values of the vectors $\mathbf{n}$ and $\mathbf{r}$. Then, by substituting the result obtained into equality (2.1), we will determine the points where there is a minimum of the function obtained with respect to the variables specifying the orientation.

Those solutions of Eq. (2.5)

$$\partial W/\partial x = H\mathbf{s}^T(\mathbf{n}\times\mathbf{h}) = 0 \tag{3.1}$$

for which the second derivative is positive, that is,

$$\partial^2 W/\partial x^2 = H\mathbf{s}^T(\mathbf{n}\times(\mathbf{s}\times\mathbf{h})) = H\mathbf{n}^T\mathbf{h} = H\cos y > 0 \tag{3.2}$$

are points of a strict minimum of the function (2.1) with respect to the angle x.

Taking account of equality (2.6), the vector $\mathbf{h}$ is determined from Eq. (3.1) as a two-valued function of the vector $\mathbf{n}$ using the formula

$$\mathbf{h} = \pm[\mathbf{n} - \mathbf{s}(\mathbf{n}^T\mathbf{s})]/\sqrt{1-(\mathbf{n}^T\mathbf{s})^2} \tag{3.3}$$

Here, condition (3.2) is satisfied for that solution of (3.3) corresponding to the plus sign. Substituting this solution into equality (2.1), we obtain the function

$$W^* = \left(-\mathbf{n}^T\mathbf{J}\mathbf{n} - 2H\sqrt{1-(\mathbf{n}^T\mathbf{s})^2} + 3\mathbf{r}^T\mathbf{J}\mathbf{r}\right)/2 \tag{3.4}$$

which depends solely on the orientation of the satellite in the orbital basis.

Using the Lagrange function with the multipliers

$$L^* = W^* + \mu_1\mathbf{n}^T\mathbf{n}/2 + \mu_2\mathbf{r}^T\mathbf{r}/2 + \mu_3\mathbf{n}^T\mathbf{r} \tag{3.5}$$

we write the equations for the stationary points of the function (3.4) in the form

$$\partial L^*/\partial \mathbf{n} = -\mathbf{J}\mathbf{n} + \mathbf{s}(\mathbf{s}^T\mathbf{n})H/\sqrt{1-(\mathbf{n}^T\mathbf{s})^2} + \mu_1\mathbf{n} + \mu_3\mathbf{r} = 0 \quad (3.6)$$

$$\partial L^*/\partial \mathbf{r} = 3\mathbf{J}\mathbf{r} + \mu_2\mathbf{r} + \mu_3\mathbf{n} = 0 \quad (3.7)$$

Since the vectors **n** and **r** are related by the conditions $\mathbf{n}^T\mathbf{n} = 1$, $\mathbf{r}^T\mathbf{r} = 1$, $\mathbf{n}^T\mathbf{r} = 0$, those solutions, for which the second differential of the Lagrange function (3.5), calculated for the set of variations $d\mathbf{r}$, $d\mathbf{n}$ connected by the equations

$$\mathbf{n}^T d\mathbf{n} = 0, \quad \mathbf{r}^T d\mathbf{r} = 0, \quad \mathbf{r}^T d\mathbf{n} + \mathbf{n}^T d\mathbf{r} = 0 \quad (3.8)$$

is a strictly positive-definite quadratic form, will be the points of a strict minimum of the function (3.4).

Calculating the second differential of function (3.5), we obtain

$$\Phi = d^2L^* = 3d\mathbf{r}^T\mathbf{J}d\mathbf{r} - d\mathbf{n}^T\mathbf{J}d\mathbf{n} + H(\mathbf{s}^T d\mathbf{n})^2[1-(\mathbf{s}^T\mathbf{n})^2]^{-3/2} +$$
$$+ \mu_1(d\mathbf{n})^2 + \mu_2(d\mathbf{r})^2 + 2\mu_3 d\mathbf{n}^T d\mathbf{r} \quad (3.9)$$

We will express the variations $d\mathbf{r}$ and $d\mathbf{n}$ of the orbital basis vectors in terms of independent variations in u, υ and w by means of the following formulae

$$d\mathbf{n} = u\mathbf{r} + \upsilon\boldsymbol{\tau}, \quad d\mathbf{r} = -u\mathbf{n} + w\boldsymbol{\tau} \quad (3.10)$$

In this case, all the equations of (3.8) will be satisfied and we shall have

$$(d\mathbf{n})^2 = u^2 + \upsilon^2, \quad (d\mathbf{r})^2 = u^2 + w^2, \quad d\mathbf{n}^T d\mathbf{r} = \upsilon w \quad (3.11)$$

We express the Lagrange multipliers from Eqs (3.6) and (3.7) using the formulae

$$\mu_1 = \mathbf{n}^T\mathbf{J}\mathbf{n} - (\mathbf{s}^T\mathbf{n})^2 H/\sqrt{1-(\mathbf{n}^T\mathbf{s})^2}$$
$$\mu_2 = -3\mathbf{r}^T\mathbf{J}\mathbf{r}, \quad \mu_3^2 = 9((\mathbf{J}\mathbf{r})^2 - (\mathbf{r}^T\mathbf{J}\mathbf{r})^2) \quad (3.12)$$

Substituting expressions (3.10)–(3.13) into equality (3.9) and taking account of the fact that it follows from Eq. (3.7) that $\boldsymbol{\tau}^T\mathbf{J}\mathbf{r} = 0$, we obtain the following quadratic form of the three variables u, υ and w

$$\Phi = 4(\mathbf{n}^T\mathbf{J}\mathbf{n} - \mathbf{r}^T\mathbf{J}\mathbf{r})u^2 + (\mathbf{n}^T\mathbf{J}\mathbf{n} - \boldsymbol{\tau}^T\mathbf{J}\boldsymbol{\tau})\upsilon^2 + 3(\boldsymbol{\tau}^T\mathbf{J}\boldsymbol{\tau} - \mathbf{r}^T\mathbf{J}\mathbf{r})w^2 +$$
$$+ H\left(\frac{(\mathbf{s}^T(\mathbf{r}u + \boldsymbol{\tau}\upsilon))^2}{(1-(\mathbf{s}^T\mathbf{n})^2)^{3/2}} - \frac{(\mathbf{s}^T\mathbf{n})^2(u^2+\upsilon^2)}{(1-(\mathbf{s}^T\mathbf{n})^2)^{1/2}}\right) - 6\mathbf{n}^T\mathbf{J}\boldsymbol{\tau}uw + 2\mu_3\upsilon w \quad (3.13)$$

The quadratic form (3.13) is written for the general case of the position of the **s** axis. For the case $\mathbf{s} = \mathbf{e}_3$ considered above, we have $\mathbf{s}^T\mathbf{n} = \sin y$ and, denoting the projections of the vectors **r** and $\boldsymbol{\tau}$ on to the $\mathbf{e}_3$ axis by r_3 and τ_3, we obtain

$$\Phi = 4(\mathbf{n}^T\mathbf{J}\mathbf{n} - \mathbf{r}^T\mathbf{J}\mathbf{r})u^2 + (\mathbf{n}^T\mathbf{J}\mathbf{n} - \boldsymbol{\tau}^T\mathbf{J}\boldsymbol{\tau})\upsilon^2 + 3(\boldsymbol{\tau}^T\mathbf{J}\boldsymbol{\tau} - \mathbf{r}^T\mathbf{J}\mathbf{r})w^2 +$$
$$+ \frac{H}{|\cos y|}\left(\frac{(r_3u + \tau_3\upsilon)^2}{\cos^2 y} - \sin^2 y(u^2+\upsilon^2)\right) - 6\mathbf{n}^T\mathbf{J}\boldsymbol{\tau}uw + 2\mu_3\upsilon w \quad (3.14)$$

Those equilibrium positions for which condition (3.2) is satisfied and the quadratic form (3.14) is strictly positive-definite will satisfy the sufficient conditions for stability. If, however, instead of condition (3.2), an inequality of the opposite sign is satisfied or there are negative characteristic numbers among the characteristic numbers of the matrix of the quadratic form (3.14), then the function (2.1) will not have a minimum (including a non-strict minimum) at the corresponding equilibrium positions.

We will now investigate the stability of the equilibrium positions found in Section 2, using equality (3.2) and the quadratic form (3.14). Since, $\mathbf{s} = \mathbf{e}_3$ in the problem considered, then, without loss of generality, the axes $\mathbf{e}_1$ and $\mathbf{e}_2$ can be numbered such that the inequality

$$A > B \quad (3.15)$$

is satisfied.

We will now consider solutions (2.19) when $k = 1$. For these solutions, condition (3.2) is satisfied for $\delta_2 = \delta_3$, that is, for the following four equilibrium positions

$$\mathbf{r} = \delta_1\mathbf{e}_3, \quad \mathbf{n} = \mathbf{h} = \delta_2\mathbf{e}_1, \quad \boldsymbol{\tau} = -\delta_1\delta_2\mathbf{e}_2 \quad (3.16)$$

For these solutions, we have $\mu_3 = 0$, $\sin y = 0$, $\tau_3 = 0$, $r_3^2 = 1$, and the quadratic form (3.14) becomes

$$\Phi = 4(A-C)u^2 + (A-B)\upsilon^2 + 3(B-C)w^2 + Hu^2 \quad (3.17)$$

The conditions for it to be positive-definite reduce to the inequalities

$$A > B, \quad B > C, \quad H > 4(C-A) \quad (3.18)$$

It follows from these inequalities that the solutions (3.16) satisfy the sufficient conditions for stability only when C is the smallest moment of inertia. The stability of these solutions holds for all values of $H > 0$.

Analysing the four solutions (2.19) in a similar manner when $k = 2$, for which condition (3.2) is satisfied when $\delta_2 = \delta_3$, we obtain that the sufficient conditions for their stability are described by the inequalities

$$B > A, \quad A > C, \quad H > 4(C - B) \tag{3.19}$$

Since the first of these contradicts condition (3.15), not one of the solutions (2.19) when $k = 2$ satisfies the sufficient conditions for stability.

Among the solutions (2.20) when $k = 1$, the following four equilibrium positions satisfy condition (3.2)

$$\mathbf{r} = \delta_1 \mathbf{e}_2, \quad \mathbf{n} = \mathbf{h} = \delta_2 \mathbf{e}_1, \quad \boldsymbol{\tau} = \delta_1 \delta_2 \mathbf{e}_3 \tag{3.20}$$

In this case, we have $\mu_3 = 0$, $\sin y = 0$, $r_3 = 0$, $\tau_3^2 = 1$, and the quadratic form (3.14) takes the form

$$\Phi = 4(A - B)u^2 + (A - C)\upsilon^2 + 3(C - B)w^2 + H\upsilon^2 \tag{3.21}$$

The conditions for it to be positive-definite reduce to the inequalities

$$A > B, \quad C > B, \quad H > C - A \tag{3.22}$$

It follows from this that the solutions (3.20) when $k = 1$ satisfy the sufficient conditions for stability when C is either the middle or the largest moment of inertia. At the same time, in the case when $A > C > B$, these solutions are stable for all values of $H > 0$ and, in the case when $C > A > B$, stability occurs for values of $H > C - B$.

A similar analysis of the solutions (2.20) when $k = 2$ leads to the conclusion that none of them satisfies the sufficient conditions for stability, since condition (3.15) is not satisfied for these solutions.

Solutions (2.21) when $k = 1$ are described by the relations

$$\mathbf{r} = \delta_1 \mathbf{e}_2, \quad \mathbf{h} = \delta_3 \mathbf{e}_1, \quad \mathbf{n} = \delta_3 \mathbf{e}_1 \cos y + \delta_2 \mathbf{e}_3 |\sin y|, \quad \cos y = H/(C - A)$$
$$\boldsymbol{\tau} = \delta_1(\mathbf{e}_3 \delta_3 \cos y - \mathbf{e}_1 \delta_2 |\sin y|), \quad H \in (0, |C - A|\,] \tag{3.23}$$

For them, we have

$$\mathbf{r}^T \mathbf{J} \mathbf{r} = B, \quad \mathbf{n}^T \mathbf{J} \mathbf{n} = A\cos^2 y + C\sin^2 y, \quad \boldsymbol{\tau}^T \mathbf{J} \boldsymbol{\tau} = A\sin^2 y + C\cos^2 y$$

$$\mathbf{n}^T \mathbf{J} \boldsymbol{\tau} = \pm(C - A)\cos y |\sin y|, \quad r_3 = 0, \quad \tau_3^2 = \cos^2 y, \quad \mu_3 = 0$$

Conditions (3.2) and (3.15) then reduce to the double inequality $C > A > B$ and the quadratic form (3.14) is written in the form

$$\Phi = \sin^2 y(C - A)\upsilon^2 + 3(A - B + (C - A)\cos^2 y)w^2 +$$
$$+ (4(A - B) + 3(C - A)\sin^2 y)u^2 \pm 6(C - A)\cos y |\sin y| uw \tag{3.24}$$

Applying Silvesters criterion, we obtain that, in the case when $C > A > B$, the quadratic form (3.24) is strictly positive-definite for all values of y apart from the points $\sin y = 0$, that is, the equilibrium position (3.23) satisfy the sufficient conditions for stability over the whole range of their existence apart from the branching point $H = C - A$.

In the case of solutions (2.21) when $k = 2$, the inequality $C > B$ follows from condition (3.2) and the quadratic form (3.14) takes the form

$$\Phi = \sin^2 y(C - B)\upsilon^2 + 3(B - A + (C - B)\cos^2 y)w^2 +$$
$$+ (4(B - A) + 3(C - B)\sin^2 y)u^2 \pm 6(C - B)\cos y |\sin y| uw \tag{3.25}$$

The condition for it to be positive-definite reduces to the inequalities

$$C > B, \quad B - A + (C - B)\cos^2 y > 0, \quad 4(B - A) + (C - B)(3 + \cos^2 y) < 0$$

which are incompatible for any value of y. Consequently, the solutions when $k = 2$ do not satisfy the sufficient conditions for stability.

The solutions (2.22) when $k = 1$ satisfy inequalities (3.2) and (3.15) in the case when $C > A > B$. For these solutions, we shall have

$$\mathbf{r}^T \mathbf{J} \mathbf{r} = A\sin^2 y + C\cos^2 y, \quad \mathbf{n}^T \mathbf{J} \mathbf{n} = A\cos^2 y + C\sin^2 y, \quad \boldsymbol{\tau}^T \mathbf{J} \boldsymbol{\tau} = B$$

$$\mathbf{n}^T \mathbf{J} \boldsymbol{\tau} = 0, \quad \tau_3 = 0, \quad r_3^2 = \cos^2 y, \quad \mu_3^2 = 9(C - A)^2 \cos^2 y \sin^2 y$$

$$H/\cos y = 4(C - A)$$

and the quadratic form (3.14) is written in the form

$$\Phi = 4\sin^2 y(C - A)u^2 + (A - B - 3(C - A)\sin^2 y)\upsilon^2 +$$
$$+ 3(B - A - (C - A)\cos^2 y)w^2 \pm 6(C - A)\cos y |\sin y| \upsilon w \tag{3.26}$$

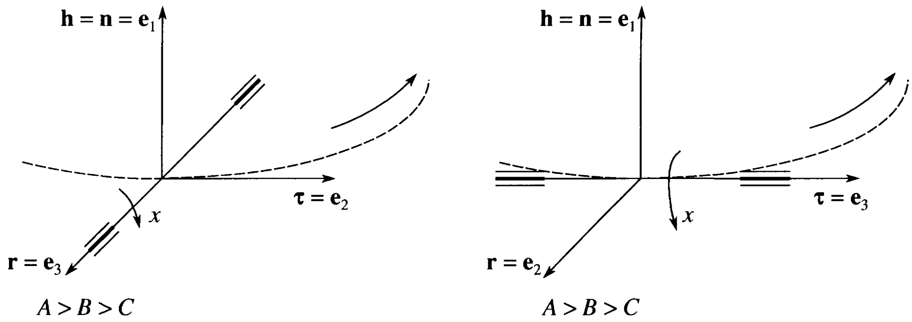

Fig. 3.

We will now calculate the sum of the two diagonal elements of the form (3.26)

$$\Phi_{vv} + \Phi_{ww} = -3(C - A) - 2(A - B) \tag{3.27}$$

Since, when $C > A > B$, expression (3.27) takes negative values, the equilibrium positions (2.22) when $k = 1$ then do not satisfy the sufficient conditions for stability. The same result is also established in a similar way for solution (2.22) when $k = 2$.

We will now investigate the stability of the equilibrium positions which correspond to the solutions of Eq. (2.17) and are described by relations (2.35), (2.40) and (2.41). It follows from formula (2.40) that these solutions only satisfy condition (3.2) when $C > A$. We will calculate the diagonal coefficient Φ_{ww} in the quadratic form (3.14). Using relation (2.39) and the equality

$$\boldsymbol{\tau}^T\mathbf{J}\boldsymbol{\tau} + \mathbf{r}^T\mathbf{J}\mathbf{r} + \mathbf{n}^T\mathbf{J}\mathbf{n} = A + B + C$$

we obtain

$$\Phi_{ww} = 3(\boldsymbol{\tau}^T\mathbf{J}\boldsymbol{\tau} - \mathbf{r}^T\mathbf{J}\mathbf{r}) = 3(A + B + C - 2\mathbf{r}^T\mathbf{J}\mathbf{r} - \mathbf{n}^T\mathbf{J}\mathbf{n}) = 9(f - C) \tag{3.28}$$

It follows from formula (2.33) and inequality (2.25) that, when $C > A$, the coefficient (3.28) takes negative values. Therefore, none of the solutions being considered satisfies the sufficient conditions for stability.

Summarizing the results of the analysis, we will list all the equilibrium positions of the system which are stable in a secular sense.

If the axis of the frame of the gyroscope is set parallel to the axis of the smallest moment of inertia ($A > B > C$), then only four solutions of (3.16) possess secular stability for all values of $H > 0$ (the left-hand side of Fig. 3).

If the axis of the frame is set parallel to the axis of the middle moment of inertia ($A > C > B$), then only four solutions of (3.20) possess secular stability for all values of $H > 0$ (the right-hand side of Fig. 3).

If the axis of the frame of the gyroscope is set parallel to the greatest moment of inertia ($C > A > B$), then only four solutions (3.20) possess secular stability in the range $H > C - A$ but eight solutions (3.23) in the range $0 < H < C - A$, which branch off from the solution (3.20) at the point $H = C - A$ (Fig. 4).

It follows from the above analysis that, for all of the stable equilibrium positions, the axis of the rotor is directed parallel to that one of the principal axes of inertia $\mathbf{e}_1$ or $\mathbf{e}_2$ with respect to which the moment of inertia is greater and the angular momentum of the characteristic rotation of the rotor has a positive projection on to the direction of the angular velocity of the orbital basis.

Note that, of the three versions of the arrangement of a gyroscope which have been considered, new stable equilibrium positions (which are different from the stable equilibrium positions of a satellite without a gyroscope) are only obtained in the case when the axis of the frame is set parallel of the axis of the greatest moment of inertia (Fig. 4). The same stable equilibrium positions can be obtained using a rotor which is set parallel to the axis of the middle moment of inertia. However, the use of a gyroscope is preferable because, when there is dissipation in the axes of the frame, damping of the oscillations of the satellite becomes possible and, in many cases, one can ensure asymptotic stability of the equilibrium positions.

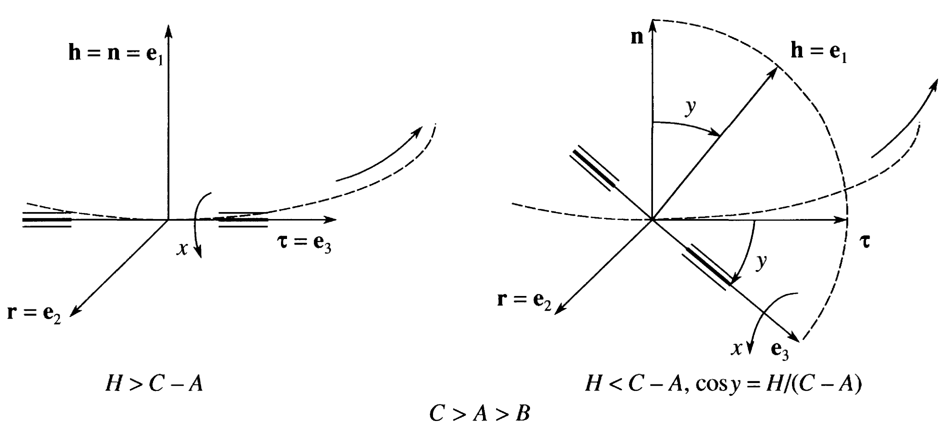

Fig. 4.

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